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The finite-size $SU(3)$ Perk–Schultz model with deformation parameter $q = \exp(2i\pi/3)$

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Abstract

From extensive numeric diagonalizations of the $SU(3)$ Perk–Schultz Hamiltonian with a special value of the anisotropy and different boundary conditions, we have observed simple regularities for a significant part of its eigenspectrum. In particular, the ground-state energy and nearby excitations belong to this part of the spectrum. Our simple formulae describing these regularities recall, apart from some selection rules, the eigenspectrum of the free-fermion model. Based on the numerical observations we formulate several conjectures. Using explicit solutions of the associated nested Bethe-ansatz equations, guessed from an analysis of the functional equations of the model, we provide evidence for some of them.

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1. Introduction

Since the pioneering work of Bethe in 1931 the Bethe ansatz and its generalizations have proved to be quite efficient tools in the description of the eigenvectors of a huge variety of one-dimensional quantum chains and two-dimensional transfer matrices (see, e.g., [1] for reviews). Models with wavefunctions given by this ansatz are considered exactly integrable. According to the Bethe ansatz the amplitudes of the wavefunctions are expressed in terms of a sum of plane waves whose wavenumbers are given in terms of non-linear and highly non-trivial coupled equations known as the Bethe-ansatz equations (BAEs). In several cases these equations, thanks to some appropriate guessing on the topology of roots, are solvable in the thermodynamic limit, providing understanding of large-distance physics.

However the exact integrability is a property independent of the lattice size and the exact solution of the associated BAE for finite chains is an important step toward the complete mathematical and physical understanding of these models. Due to the complexity of the BAE, to our knowledge, only in two special cases are some of the solutions known analytically, namely, the trivial free-fermion case and the XXZ chain at the special anisotropy $\Delta = -1/2$ [2,3]. The solution in this last case is obtained by exploring the functional relations of the model [3]. Even in the last case, although several exact properties of the wavefunction have been conjectured [4], a complete and closed calculation of their amplitudes is still missing. In this paper we are going to present a new set of analytical solutions of BAEs for finite chains. These solutions correspond to BAEs of the anisotropic $SU(3)$ Perk–Schultz model [5], or the anisotropic $SU(3)$ Sutherland model [6], at a special value of the anisotropy. In contrast to the XXZ case, the Bethe ansatz for this model is of nested type and the solutions will be derived by generalizing the functional method originally applied to the XXZ chain.

The paper is organized as follows. In section 2 we give the main definitions and formulate the corresponding BAE. In section 3 we state a set of conjectures that were obtained from extensive 'experimental' work on exact brute-force diagonalizations of the quantum chains. In section 4 we derive, for the Hamiltonian with periodic boundary conditions, the functional relations, and at a special value of the anisotropy some solutions for the eigenspectra are derived. In section 5 we present and test directly a set of solutions of the BAEs, and explain partially the conjectures announced in section 3. Finally in section 6 we present our conclusions and a summary of our results.

2. The $SU(3)$ Perk–Schultz model

The $SU(3)$ Perk–Schultz model [5] is the anisotropic version of the $SU(3)$ Sutherland model [6], with the Hamiltonian, in an L-site chain, given by

$$H_q^p = \sum_{j=1}^{L-1} H_{j,j+1} + pH_{L,1} \qquad (p = 0, 1),$$

where

$$H_{i,j} = -\sum_{a=0}^{1}\sum_{b=a+1}^{2} (E_i^{ab}E_j^{ba} + E_i^{ba}E_j^{ab} - qE_i^{aa}E_j^{bb} - 1/qE_i^{bb}E_j^{aa}). \tag{1}$$

The 3×3 matrices E^{ab} have elements $(E^{ab})_{cd} = \delta_c^a\delta_d^b$ and $q = \exp(\mathrm{i}\eta)$ is the anisotropy of the model. The cases of free and periodic boundary conditions are obtained by setting $p = 0$ and $p = 1$ in (1), respectively. This Hamiltonian describes the dynamics of a system containing three classes of particles (0, 1, 2) with on-site hard-core exclusion. At $q = 1$ the model is $SU(3)$ symmetric and for $q \neq 1$ the model has a $U(1) \bigotimes U(1)$ symmetry due to the conservation of the number of particles of each species. Consequently we can separate the Hilbert space into block disjoint sectors labelled by (n_0, n_1, n_2), where $n_i = 0, 1, \ldots, L$ is the number of particle of species i $(i = 0, 1, 2)$. The Hamiltonian has an S_3 symmetry due to its invariance under the permutation of distinct species, that implies that all the energies can be obtained from the sectors (n_0, n_1, n_2), where $n_0 \leqslant n_1 \leqslant n_2$ and $n_0 + n_1 + n_2 = L$. Moreover in the special case of free boundaries $(p = 0)$, the quantum chain (1) is also invariant under the additional quantum $SU(3)_q$ symmetry implying that all energies in the sector (n_0', n_1', n_2') with $n_0' \leqslant n_1' \leqslant n_2'$ are degenerate with the energies belonging to the sectors (n_0, n_1, n_2) with $n_0 \leqslant n_1 \leqslant n_2$, if $n_0' \leqslant n_0$ and $n_0' + n_1' \leqslant n_0 + n_1$.

The eigenenergies of the Hamiltonian (1) for $p = 0$ or 1 in the sector (n_0, n_1, n_2) are given by

$$E = -\sum_{j=1}^{n_0+n_1}\left(-q - \frac{1}{q} + \frac{\sin(u_j - \eta/2)}{\sin(u_j + \eta/2)} + \frac{\sin(u_j + \eta/2)}{\sin(u_j - \eta/2)}\right), \tag{2}$$

where the variables $\{u_j, j = 1, 2, \ldots, n_0 + n_1\}$ and the auxiliary variables $\{v_j, j = 1, 2, \ldots, n_0\}$ are the roots of the coupled Bethe ansatz. These BAEs are of nested type and in the case of periodic boundary they are given by (see e.g. [7, 8])

$$\begin{gathered}\left[\frac{\sin(u_k+\eta/2)}{\sin(u_k-\eta/2)}\right]^L = -\prod_{i=1}^{n_0+n_1}\frac{\sin(u_k-u_i+\eta)}{\sin(u_k-u_i-\eta)}\prod_{j=1}^{n_0}\frac{\sin(u_k-v_j-\eta/2)}{\sin(u_k-v_j+\eta/2)},\\ \prod_{i=1}^{n_0}\frac{\sin(v_l-v_i+\eta)}{\sin(v_l-v_i-\eta)}\prod_{j=1}^{n_0+n_1}\frac{\sin(v_l-u_j-\eta/2)}{\sin(v_l-u_j+\eta/2)} = -1,\end{gathered} \tag{3}$$

where $k = 1, \ldots, n_0 + n_1$ and $l = 1, \ldots, n_0$.

In the case of free boundary the BAEs are given by [9]

$$\begin{gathered}\left[\frac{\sin(u_k+\eta/2)}{\sin(u_k-\eta/2)}\right]^{2L}\prod_{i=1}^{n_0}\frac{\sin(u_k+v_i+\eta/2)\sin(u_k-v_i+\eta/2)}{\sin(u_k+v_i-\eta/2)\sin(u_k-v_i-\eta/2)}\\ = \prod_{j=1,j\neq k}^{n_0+n_1}\frac{\sin(u_k+u_j+\eta)\sin(u_k-u_j+\eta)}{\sin(u_k+u_j-\eta)\sin(u_k-u_j-\eta)},\\ \prod_{i=1,i\neq l}^{n_0}\frac{\sin(v_l+v_i+\eta)\sin(v_l-v_i+\eta)}{\sin(v_l+v_i-\eta)\sin(v_l-v_i-\eta)} = \prod_{j=1}^{n_0+n_1}\frac{\sin(v_l+u_j+\eta/2)\sin(v_l-u_j+\eta/2)}{\sin(v_l+u_j-\eta/2)\sin(v_l-u_j-\eta/2)},\end{gathered} \tag{4}$$

where $k = 1, \ldots, n_0+n_1$ and $l = 1, \ldots, n_0$. In the case of periodic boundaries the momentum $P = 2\pi l/L$ $(l = 0, 1, \ldots, L-1)$ of the associated eigenstate is given by

$$\exp(iP) = \prod_{k=1}^{n_0+n_1}\frac{\sin(u_k-\eta/2)}{\sin(u_k+\eta/2)}. \tag{5}$$

The solutions of the BAEs will provide the eigenenergies of (1) if they correspond to non-zero norm Bethe states. The combinatory nature of the Bethe wavefunctions implies that solutions of (3) or (4) with coinciding roots produce null states. Nevertheless the requirement of non-coinciding roots does not necessarily ensure a genuine eigenvector, since conspicuous cancellation, even in this case, can render a vector with null norm. Although all eigenenergies of the Hamiltonian can be obtained, apart from predicted degeneracies, by restricting to the sectors (n_0, n_1, n_2) with $n_0 \leqslant n_1 \leqslant n_2$, the Bethe ansatz implementation in its coordinate version is valid for arbitrary values of n_0, n_1, n_2. However, as we shall see in section 5, several solutions with non-coinciding roots for sectors out of the range $n_0 \leqslant n_1 \leqslant n_2$, but corresponding to the null state, can be obtained. In fact even in the XXZ chain, where the BAEs are simpler, solutions with non-coinciding roots[3] that correspond to null-norm states can be obtained when the number of roots n is out of the range $n \leqslant L/2$. In the case of the BAE for the XXZ chain, in a recent paper [11] Baxter gives strong evidence that with suitable parametrization the entire eigenspectra can be obtained from the non-coinciding roots of the associated BAE in the sector with the number of roots $n \leqslant L/2$. In a similar way we are going to assume in this paper that all distinct eigenenergies of (1) can be obtained from the solutions $\{u_i\}, \{v_i\}$, with non-coinciding values in the sectors when $n_0 \leqslant n_1 \leqslant n_2$. Conversely, solutions in sectors out of this range should necessarily be degenerate with energies occurring in sectors within the range, if they do not correspond to zero-norm states.

3. Conjectures merged from numerical studies

In this section we state a series of conjectures that are consistent with the exact brute-force diagonalization of the Hamiltonian (1) with free $(p = 0)$ and periodic $(p = 1)$ boundary

[3] We have found, for example, a continuous set of non-coinciding roots of the BAE for the periodic XXZ with $L = 4$ sites and $n = 3$. All this set of solutions give us zero eigenvectors (see also [10] for further considerations).

condition at $q = \exp(2i\pi/3)$. Some of these conjectures will be proved in the following sections. Let us consider separately the case of periodic and free boundaries.

3.1. Periodic chain

Conjecture 1. *The Hamiltonian (1) with L sites at* $q = \exp(2i\pi/3)$ *has eigenvectors (not all of them) with energy and momentum given by*

$$E_I = -\sum_{j\in I}\left(1 + 2\cos\frac{2\pi j}{L}\right), \tag{6}$$

$$P_I = \frac{2\pi}{L}\sum_{j\in I} j, \tag{7}$$

with I being a subset of $\mathcal{I}$ *unequal elements of the set* $\{1, 2, \ldots, L\}$*. The number* $\mathcal{I}$ *has to be odd,* $\mathcal{I} = 2k+1$*, and the sector of appearance of the above levels is* $S_k \equiv (k, k+1, L-2k-1)$*,* $0 \leqslant k \leqslant (L-1)/2$*.*

The lowest eigenenergy among the above conjectured values (6) is obtained for the particular set $I_0^{(k)} = \{1, 2, \ldots, k\} \cup \{L-k, \ldots, L\}$, since in this case all contributions $-[1 + 2\cos(2\pi j/L)]$ to (6) have the lowest possible values. The corresponding eigenstate has zero momentum and energy given by

$$E_0^{(k)} = -\sum_{j\in I_0^{(k)}}\left(1 + 2\cos\frac{2\pi j}{L}\right) = -2k - 1 - 2\frac{\sin(\pi(2k+1)/L)}{\sin(\pi/L)}. \tag{8}$$

Conjecture 2. *For arbitrary* $L = 3n + l$ $(l = 1, 2, 3)$*, the eigenenergy* $E_0^{(n)}$ *is the lowest one in the sector* S_n*; moreover, if* $l \neq 3$ $(L \neq 3n)$*, it is the ground-state energy of the model.*

3.2. Free boundaries

In order to state our conjectures let us define again the special set of sectors of the Hamiltonian (1) with $p = 0$:

$$S_k = \left(\mathrm{Int}\left(\frac{k}{2}\right), \mathrm{Int}\left(\frac{k+1}{2}\right), L-k\right), \qquad k = 0, 1, \ldots, L. \tag{9}$$

Due to the quantum symmetry $SU(3)_q$, distinct sectors show the same eigenenergies. For example, for $L = 7$ the sectors are

$$\begin{array}{ll} S_0 = (0,0,7) & S_1 = (0,1,6) \\ S_2 = (1,1,5) & S_3 = (1,2,4) \\ S_4 = (2,2,3) & S_5 = (2,3,2) \\ S_6 = (3,3,1) & S_7 = (3,4,0) \end{array}$$

and we have a special ordering

$$S_0 \subset S_1 \subset S_2 \subset S_3 \subset S_4 \equiv S_5 \supset S_6 \supset S_7. \tag{10}$$

This means that, for example, all eigenvalues in sector S_2 can also be found in sectors S_3, S_4 and S_5, and on the other hand all eigenvalues in sector S_7 also appear in sectors S_6, S_5 and S_4. Sectors S_4 and S_5 are totally equivalent. In this example, let us call the sectors S_0, S_1, S_2, S_3, S_4 the left sectors and S_5, S_6, S_7 the right ones. This can be directly generalized to any $L = 3n+1$ or $L = 3n + 2$, obtaining $L - n$ left sectors and $n + 1$ right ones. In the case where $L = 3n$

the sectors S_k with $k = 0, 1, \ldots, 2n-1$ and $k = 2n+1, \ldots, L$ are the left and right sectors, respectively. The sector $S_{2n} = (n, n, n)$ is degenerate with two sectors $S_{2n-1} = (n-1, n, n+1)$ and $S_{2n+1} = (n, n+1, n-1)$ ($S_{2n-1} \equiv S_{2n+1}$) and can be considered as either a left or right sector. We now state the conjecture.

Conjecture 3. *Let $L = 3n + l$ ($l = 0, 1, 2$). Then the Hamiltonian (1) with free boundaries ($p = 0$ at $q = \exp(2i\pi/3)$ has eigenvectors with energies given by*

$$E_I = -\sum_{j\in I}\left(1 + 2\cos\frac{\pi j}{L}\right), \tag{11}$$

with I an arbitrary subset formed by k distinct elements of the set $\{1, 2, \ldots, L-1\}$. Moreover, if S_k is a left sector then these eigenvalues appear in the sectors $S_k, S_{k+1}, \ldots, S_{L-n}$ (S_{L-n+1} for $l = 0$), and if S_k is a right sector the eigenvalues appear in the sectors $S_{L-n-1}, S_{L-n}, \ldots, S_{k+1}$.

As a consequence of conjecture 3 the Hamiltonian (1) has the special eigenvalues

$$E^{(k)} = -\sum_{j=1}^{k}\left(1 + 2\cos\frac{\pi j}{L}\right) = 1 - k - \frac{\sin(\pi(2k+1)/2L)}{\sin(\pi/2L)} \tag{12}$$

and we are now in a position to formulate a remarkable conjecture.

Conjecture 4. *The lowest energy in the sector S_k is $E^{(k)}$ or $E^{(k-1)}$ if S_k is a left or a right sector respectively.*

The minimal value of $E^{(k)}$ is obtained for $k = L - n - 1$ and our 'numerical experiments' induce the conjecture:

Conjecture 5. *The ground-state energy of the Hamiltonian (1) with free boundary at $q = \exp(2i\pi/3)$ is given by*

$$E_0 = E^{(L-n-1)} = 2 - L + n - \frac{\sin(\pi(2n+1)/2L)}{\sin(\pi/2L)}. \tag{13}$$

4. Functional relations for the anisotropic $SU(3)$ Perk–Schultz model

We are going to obtain analytically some of the conjectured results presented in the previous section. Let us consider initially the periodic case when $p = 1$ in the Hamiltonian (1). The eigenenergies in the sectors with 'particle numbers' (n_0, n_1, n_2) are given by (2) where the Bethe roots $\{u_j, j = 1, 2, \ldots, n_0 + n_1 \equiv m_2\}$ and $\{v_j, j = 1, 2, \ldots, n_0 \equiv m_1\}$ are obtained by solving the BAEs (3). Below, to simplify the notation, we write $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$ instead of v_j and u_j, respectively.

Defining the pair of sine polynomials

$$Q_l(\lambda) = \prod_{j=1}^{m_l}\sin(\lambda - \lambda_j^{(l)}), \qquad l = 1, 2, \tag{14}$$

the BAEs (3) can be rewritten as

$$Q_1(\lambda_j^{(1)} + \eta)Q_2(\lambda_j^{(1)} - \eta/2) + Q_1(\lambda_j^{(1)} - \eta)Q_2(\lambda_j^{(1)} + \eta/2) = 0 \qquad (j = 1, 2, \ldots, m_1), \tag{15}$$

$$\sin^L(\lambda_k^{(2)} + \eta/2)Q_1(\lambda_k^{(2)} + \eta/2)Q_2(\lambda_k^{(2)} - \eta) + \sin^L(\lambda_k^{(2)} - \eta/2) \times Q_1(\lambda_k^{(2)} - \eta/2)Q_2(\lambda_k^{(2)} + \eta) = 0 \qquad (k = 1, 2, \ldots, m_2). \tag{16}$$

Since from the definitions (14) $Q_l(\lambda_j^{(l)}) = 0$ for any Bethe roots $\lambda_j^{(l)}$ ($l = 1, 2$), we should have the functional relations

$$Q_1(\lambda+\eta)Q_2(\lambda-\eta/2) + Q_1(\lambda-\eta)Q_2(\lambda+\eta/2) = T_2(\lambda)Q_1(\lambda), \tag{17}$$

$$\sin^L(\lambda+\eta/2)Q_1(\lambda+\eta/2)Q_2(\lambda-\eta) + \sin^L(\lambda-\eta/2)Q_1(\lambda-\eta/2)Q_2(\lambda+\eta) = T_1(\lambda)Q_2(\lambda), \tag{18}$$

where $T_2(\lambda)$ and $T_1(\lambda)$ are unknown sine polynomials of order m_2 and $L + m_1$, respectively. Shifting $\lambda \to \lambda \mp \eta/2$ in (17) and inserting the result in (18) we obtain

$$\begin{aligned}\sin^L(\lambda \mp \eta/2)Q_2(\lambda \pm \eta) + \sin^L(\lambda \pm \eta/2)T_2(\lambda \mp \eta/2)\}Q_1(\lambda \mp \eta/2) \\ = \sin^L(\lambda \pm \eta/2)Q_1(\lambda \mp 3\eta/2) + T_1(\lambda)\}Q_2(\lambda).\end{aligned} \tag{19}$$

We now suppose that $Q_1(\lambda \pm \eta/2)$ and $Q_2(\lambda)$ have no common roots; in this case

$$\sin^L(\lambda \mp \eta/2)Q_2(\lambda \pm \eta) + \sin^L(\lambda \pm \eta/2)T_2(\lambda \mp \eta/2)\} = T^{\pm}(\lambda)Q_2(\lambda), \tag{20}$$

$$\sin^L(\lambda \pm \eta/2)Q_1(\lambda \mp 3\eta/2) + T_1(\lambda) = T^{\pm}(\lambda)Q_1(\lambda \mp \eta/2), \tag{21}$$

where $T^{\pm}(\lambda)$ are sine polynomials[4] of degree L. The subtraction of equations (21) among themselves gives us

$$\begin{aligned}\sin^L(\lambda+\eta/2)Q_1(\lambda-3\eta/2) - T^+(\lambda)Q_1(\lambda-\eta/2) \\ + T^-(\lambda)Q_1(\lambda+\eta/2) - \sin^L(\lambda-\eta/2)Q_1(\lambda+3\eta/2) = 0.\end{aligned} \tag{22}$$

Similarly both equations (20) give us the relation

$$\begin{aligned}\sin^L(\lambda)\sin^L(\lambda+\eta)Q_2(\lambda-3\eta/2) - \sin^L(\lambda+\eta)T^-(\lambda-\eta/2)Q_2(\lambda-\eta/2) \\ + \sin^L(\lambda-\eta)T^+(\lambda+\eta/2)Q_2(\lambda+\eta/2) \\ - \sin^L(\lambda)\sin^L(\lambda-\eta)Q_2(\lambda+3\eta/2) = 0.\end{aligned} \tag{23}$$

Up to now our relations are valid for arbitrary values of the anisotropy η and we now are going to restrict to the particular case $\eta = 2\pi/3$ ($q = \exp(2\mathrm{i}\pi/3)$, where the several conjectures announced in section 3 were expected to be valid. At this special value of the anisotropy we have the symmetry

$$Q_l(\lambda-3\eta/2) = Q_l(\lambda-\pi) = Q_l(\lambda+\pi) = Q_l(\lambda+3\eta/2) \qquad l = 1, 2, \tag{24}$$

and equations (22) and (23) are given by

$$\phi(\lambda)Q_1(\lambda-\pi) - T^+(\lambda)Q_1(\lambda-\pi/3) + T^-(\lambda)Q_1(\lambda+\pi/3) = 0, \tag{25}$$

and

$$\begin{aligned}-\sin^L(\lambda)\phi(\lambda-\pi)Q_2(\lambda-\pi) - \sin^L(\lambda+2\pi/3)T^-(\lambda-\pi/3)Q_2(\lambda-\pi/3) \\ + \sin^L(\lambda-2\pi/3)T^+(\lambda+\pi/3)Q_2(\lambda+\pi/3) = 0,\end{aligned} \tag{26}$$

where

$$\phi(\lambda) = \sin^L(\lambda+\pi/3) - \sin^L(\lambda-\pi/3). \tag{27}$$

The shift $\lambda \to \lambda \pm 2\pi/3$ in (25) and (26) shows that these equations are equivalent to the linear matrix equations

$$\left|\begin{matrix} \phi(\lambda) & -T^+(\lambda) & T^-(\lambda) \\ T^-(\lambda+2\pi/3) & \phi(\lambda+2\pi/3) & -T^+(\lambda+2\pi/3) \\ -T^+(\lambda-2\pi/3) & T^-(\lambda-2\pi/3) & \phi(\lambda-2\pi/3) \end{matrix}\right| \left|\begin{matrix} Q_1(\lambda-\pi) \\ Q_1(\lambda-\pi/3) \\ Q_1(\lambda+\pi/3) \end{matrix}\right| = 0, \tag{28}$$

[4] These polynomials are the eigenvalues of the transfer matrices corresponding to the fundamental representations of $SU(3)$ in the auxiliary space.

and

$$\begin{vmatrix} \phi(\lambda-\pi) & T^-(\lambda-\pi/3) & -T^+(\lambda+\pi/3) \\ -T^+(\lambda-\pi) & -\phi(\lambda-\pi/3) & T^-(\lambda+\pi/3) \\ T^-(\lambda-\pi) & -T^+(\lambda-\pi/3) & \phi(\lambda+\pi/3) \end{vmatrix} \begin{vmatrix} \tilde{Q}_2(\lambda-\pi) \\ \tilde{Q}_2(\lambda-\pi/3) \\ \tilde{Q}_2(\lambda+\pi/3) \end{vmatrix} = 0 \tag{29}$$

respectively. In (29) we defined the new function $\tilde{Q}_2(\lambda) = \sin^L(\lambda) Q_2(\lambda)$. It is clear that $T_2(\lambda+\pi) = T_2(\lambda-\pi)$ and consequently from (20) $T_\pm(\lambda+\pi) = T_\pm(\lambda-\pi)$. Equations (28) and (29) imply that non-trivial solutions are obtained if the determinants of the matrices appearing in these equations vanish. Actually, by shifting $\lambda \to \lambda+\pi$ in the determinant from (29) we clearly see that this last determinant vanishes if that from (28) also vanishes.

The calculation of the general solutions $T^\pm(\lambda)$ that render a null determinant is a quite difficult task; however, simple solutions can be obtained (rank 1) by imposing a proportionality between the columns of the matrix generating the determinant[5], i.e.

$$\begin{aligned} \frac{\phi(\lambda)}{-T^+(\lambda)} &= \frac{T^-(\lambda+2\pi/3)}{\phi(\lambda+2\pi/3)} = \frac{-T^+(\lambda-2\pi/3)}{T^-(\lambda-2\pi/3)}, \\ \frac{-T^+(\lambda)}{T^-(\lambda)} &= \frac{\phi(\lambda+2\pi/3)}{-T^+(\lambda+2\pi/3)} = \frac{-T^-(\lambda-2\pi/3)}{\phi(\lambda-2\pi/3)}. \end{aligned} \tag{30}$$

We can verify that the above relations are equivalent to the independent equations

$$T^+(\lambda)T^-(\lambda+2\pi/3) = -\phi(\lambda)\phi(\lambda+2\pi/3), \tag{31}$$

$$T^+(\lambda)T^+(\lambda-2\pi/3) = \phi(\lambda)T^-(\lambda-2\pi/3). \tag{32}$$

In order to find solutions of these last equations, it will be useful to use the general relation

$$a^L - b^L = \prod_{j=1}^{L}(a-\omega^j b), \qquad \omega = \exp(2\pi i/L) \tag{33}$$

to write

$$\phi(\lambda) = \sin^L(\lambda+\pi/3) - \sin^L(\lambda-\pi/3) = \prod_{l=1}^{L} f_l(\lambda), \tag{34}$$

where

$$f_l(\lambda) = \sin(\lambda+\pi/3) - \omega^l \sin(\lambda-\pi/3) \qquad (l = 1, \ldots, L). \tag{35}$$

Now consider any subset I of non-repeated integers of the set $I_0 = \{1, 2, \ldots, L\}$, and the complementary subset $\bar{I}$, such that $I \bigoplus \bar{I} = I_0$. We may try to solve (31) and (32) by the ansatz

$$T^\pm(\lambda) = t_0^\pm \prod_{l\in I} f_l(\lambda \pm 2\pi/3) \prod_{m\in\bar{I}} f_m(\lambda), \tag{36}$$

where $t_0^\pm$ are unknown constants. This ansatz implies

$$T^+(\lambda)T^-(\lambda+2\pi/3) = t_0^+ t_0^- \prod_{l=1}^{L} f_l(\lambda+2\pi/3) \prod_{m=1}^{L} f_m(\lambda) = t_0^+ t_0^- \phi(\lambda+2\pi/3)\phi(\lambda), \tag{37}$$

where (34) was used in the last equality. Equation (31) implies the constraint

$$t_0^+ t_0^- = -1. \tag{38}$$

[5] The idea to consider decreased rank in the functional relations was used previously in [12] to explain simple energy levels of a special case of the XXZ chain.

Also from (35) and (33)

$$T^+(\lambda)T^+(\lambda-2\pi/3) = (t_0^+)^2\phi(\lambda)\prod_{l\in I} f_l(\lambda+2\pi/3)\prod_{m\in\bar I} f_m(\lambda-2\pi/3)$$
$$= (t_0^+)^2\phi(\lambda)T^-(\lambda-2\pi/3)/t_0^-, \tag{39}$$

and (32) implies, by using (38), that

$$(t_0^-)^3 = 1, \qquad t_0^+ = -1/t_0^-. \tag{40}$$

Then the ansatz (35) with (38) gives us a set of solutions for $T^\pm(\lambda)$, that when inserted in the matrix equations (28) and (29) will provide the functions $Q_1(\lambda)$ and $Q_2(\lambda)$. The zeros of these last functions are the Bethe-ansatz roots and the eigenenergies are calculated by using in (2) the roots of $Q_2(\lambda)$. Instead of calculating the energies through this procedure, we are going to calculate them using the transfer matrix eigenvalues $T^-(\lambda)$. From (2) and the definition of $Q_2(\lambda)$ it is not difficult to obtain the relation

$$E = \frac{\sqrt{3}}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\left(\frac{Q_2(\lambda)}{Q_2(-\lambda)}\right)\bigg|_{\lambda=\pi/3}. \tag{41}$$

On the other hand let us expand (23) with $\eta = 2\pi/3$ for $\lambda = \eta+\epsilon$, $\epsilon \ll 1$. The terms of the lowest order give us the relation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\left(\frac{Q_2(\lambda)}{Q_2(-\lambda)}\right)\bigg|_{\lambda=\pi/3} = -\frac{L}{\sqrt{3}} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\ln T^-(\lambda)|_{\lambda=\pi/3}, \tag{42}$$

that from (41) provides the simple result

$$E = -\frac{L}{2} - \frac{\sqrt{3}}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln T^-(\lambda)|_{\lambda=\pi/3}. \tag{43}$$

The eigenenergies associated with our solutions $T^-(\lambda)$ are then obtained by inserting (36) in (43), and we obtain after some simple algebraic manipulations

$$E = -L + \sum_{m\in\bar I}\left(1+2\cos\left(\frac{2\pi m}{L}\right)\right) = -\sum_{l\in I}\left(1+2\cos\left(\frac{2\pi l}{L}\right)\right), \tag{44}$$

where we used the formula

$$\sum_{l\in I\cup\bar I}(1+2\cos(2\pi l/L)) = \sum_{l=1}^{L}(1+2\cos(2\pi l/L)) = L. \tag{45}$$

Also the zero-order term in the same expansion of (23) gives us

$$\frac{T^-(\pi/3)}{\sin^L(2\pi/3)} = \frac{Q_2(\pi/3)}{Q_2(-\pi/3)} = \prod_{k=1}^{m_2}\frac{\sin(u_k-\eta/2)}{\sin(u_k+\eta/2)} = \exp(\mathrm{i}P), \tag{46}$$

where from (5) P is the momentum associated with our solution $T^-(\lambda)$ given in (36). Inserting (36) into (46) we obtain after some simple calculations

$$\exp(\mathrm{i}P) = (-1)^{L+1}t_0^-\exp\left(-\frac{2\pi\mathrm{i}}{L}\sum_{m\in\bar I}m\right) = t_0^-\exp\left(\frac{2\pi\mathrm{i}}{L}\sum_{l\in I}l\right), \qquad (t_0^-)^3 = 1. \tag{47}$$

5. Analytic solutions of the Bethe-ansatz equations

As we discussed in the previous section, at least in the periodic case, the Bethe-ansatz roots, corresponding to the eigenenergies (44) we observed, can be obtained from the expansion of $Q_2(\lambda)$ given in (14), derived by solving (29) with $T^{\pm}(\lambda)$ given by the ansatz (36). Distinctly in this section we are going to present in a direct way a set of guessed solutions $\{u_i, v_j\}$ of the BAEs that gives the energies conjectured in section 3. We show that they are correct by a direct substitution into the BAEs. We present solutions of the BAEs for the periodic and free-boundary cases. As we conjectured in section 3, in the case of periodic boundaries there exist some selection rules in the spectrum composition (see conjecture 1). At the end of this section we are going to explain partially this conjecture.

Let us consider separately the periodic and free-boundary cases.

5.1. Periodic case

The BAEs (3) at $\eta = 2\pi/3$, expressed in terms of the variables

$$x_k = \frac{\sin(u_k - \pi/3)}{\sin(u_k + \pi/3)}, \qquad y_l = \frac{\sin(v_l - \pi/3)}{\sin(v_l + \pi/3)}, \tag{48}$$

with $k = 1, 2, \ldots, n_0 + n_1$ and $l = 1, 2, \ldots, n_0$, are given by

$$(-1)^{n_1+1} \prod_{j'=1}^{n_0} \frac{1 + y_j + y_j y_{j'}}{1 + y_{j'} + y_j y_{j'}} \prod_{k'=1}^{n_0+n_1} \frac{1 + y_j + y_j x_{k'}}{1 + x_{k'} + y_j x_{k'}} = 1, \qquad (j = 1, 2, \ldots, n_0) \tag{49}$$

and

$$(-1)^{n_1+1} \prod_{j'=1}^{n_0} \frac{1 + x_k + x_k y_{j'}}{1 + y_{j'} + x_k y_{j'}} \prod_{k'=1}^{n_0+n_1} \frac{1 + x_k + x_k x_{k'}}{1 + x_{k'} + x_k x_{k'}} = x_k^L \qquad (k = 1, 2, \ldots, n_0 + n_1). \tag{50}$$

Let us fix $2n_0 + n_1 = L$. Our guessed solutions are obtained by considering $\{x_h, y_l\}$ $(k = 1, \ldots, n_0 + n_1, l = 1, \ldots, n_0)$ as an arbitrary permutation of $\{\omega, \omega^2, \ldots, \omega^L\}$, where $\omega = \exp(2\pi\mathrm{i}/L)$. In this case, the left-hand side of equation (49) takes the form

$$(-1)^{L+1} \prod_{l=1}^{L} \frac{1 + y_j + y_j \omega^l}{1 + \omega^l + y_j \omega^l}. \tag{51}$$

Using the identity (33) and the fact that $y_j^L = 1$ we can rewrite this product as

$$(-1)^{L+1} \frac{(1 + y_j)^L + (-1)^{L+1} y_j^L}{1 + (-1)^{L+1}(1 + y_j)^L} = 1. \tag{52}$$

It is evident that the second BAE is also satisfied due to equality $x_k^L = 1$.

Consequently we have found a subclass of solutions for the nested BAEs. These solutions are characterized by the subset I with unequal elements of the set $I_0 = \{1, 2, \ldots, L\}$, and have the energy

$$E_I = -\sum_{k=1}^{n_0+n_1} (1 + x_k + x_k^{-1}) = -\sum_{l \in I} (1 + 2\cos(2\pi l/L)) \tag{53}$$

and momentum

$$P_I = \sum_{k=1}^{n_0+n_1} \frac{1}{\mathrm{i}} \ln(x_k) = \frac{2\pi}{L} \sum_{l \in I} l. \tag{54}$$

Comparing the above relations with relations (6) and (7) we verify that our guessed solutions are consistent with conjecture 1. It is not clear however whether the corresponding wavefunction is not a zero vector.

5.2. Free-boundary case

The BAEs (4) at $\eta = 2\pi/3$, expressed in terms of the same variables x_k and y_l with $k = 1, 2, \ldots, n_0 + n_1$ and $l = 1, 2, \ldots, n_0$, are given by

$$\prod_{j'=1, j'\neq j}^{n_0} \left(\frac{1 + y_j + y_j y_{j'}}{1 + y_{j'} + y_j y_{j'}}\right)\left(\frac{y_j + y_{j'} + y_j y_{j'}}{1 + y_j + y_{j'}}\right)\prod_{k'=1}^{n_0+n_1}\left(\frac{1 + y_j + y_j x_{k'}}{1 + x_{k'} + y_j x_{k'}}\right)\left(\frac{y_j + x_{k'} + y_j x_{k'}}{1 + y_j + x_{k'}}\right) = 1$$
$$(j = 1, 2, \ldots, n_0) \tag{55}$$

and

$$\prod_{j'=1}^{n_0}\left(\frac{1 + x_k + x_k y_{j'}}{1 + y_{j'} + x_k y_{j'}}\right)\left(\frac{x_k + y_{j'} + x_k y_{j'}}{1 + x_k + y_{j'}}\right)\prod_{k'=1, k'\neq k}^{n_0+n_1}\left(\frac{1 + x_k + x_k x_{k'}}{1 + x_{k'} + x_k x_{k'}}\right)\left(\frac{x_k + x_{k'} + x_k x_{k'}}{1 + x_k + x_{k'}}\right) = x_k^{2L}$$
$$(k = 1, 2, \ldots, n_0 + n_1). \tag{56}$$

Now let us fix $2n_0 + n_1 = L - 1$. Our guessed solutions are now given by the set $\{x_k, y_l\}$ $(k = 1, \ldots, n_0+n_1; l = 1, \ldots, n_0)$ formed by an arbitrary permutation of $\{\omega, \omega^2, \ldots, \omega^{L-1}\}$, where $\omega = \exp(\mathrm{i}\pi/L)$. Using the identity

$$\prod_{m=1}^{L-1}\frac{(a + \omega^m)(1/a + \omega^m)}{(b + \omega^m)(1/b + \omega^m)} = \frac{b^{L-1}}{a^{L-1}}\frac{(b^2 - 1)}{(a^2 - 1)}\frac{(a^{2L} - 1)}{(b^{2L} - 1)}, \tag{57}$$

and the fact that $y_i^L = 1$ we can easily verify that the BAEs (55) and (56) are satisfied.

As in the periodic case, we have found a subclass of solutions for the nested BAEs. These solutions are characterized by a subset $I \subset \{1, 2, \ldots, L - 1\}$ and have the energy

$$E_I = -\sum_{k=1}^{n_0+n_1}(1 + x_k + x_k^{-1}) = -\sum_{l\in I}(1 + 2\cos(\pi l/L)). \tag{58}$$

All these solutions are consistent with conjecture 3, so we think that corresponding Bethe wavefunction is not a zero vector.

Finally, in order to conclude this section, we intend to explain partially the selection rules formulated in conjecture 1 for the periodic case. We are going to do this by exploiting our solutions (36) for $T^{\pm}(\lambda)$ of the functional relations of section 4 with the help of some ideas developed in the papers [10].

Inserting our solutions (36) for $T^{\pm}(\lambda)$ into equation (25) we obtain

$$\prod_{l=1}^{L} f_l(\lambda)Q_1(\lambda - \pi) - t_0^+\prod_{l\in I} f_l(\lambda + 2\pi/3)\prod_{m\in\bar{I}} f_m(\lambda)Q_1(\lambda - \pi/3)$$
$$+ t_0^-\prod_{l\in I} f_l(\lambda - 2\pi/3)\prod_{m\in\bar{I}} f_m(\lambda)Q_1(\lambda + \pi/3) = 0. \tag{59}$$

Dividing by the common factor $\prod_{m\in\bar{I}} f_m(\lambda)$ we obtain

$$F_1(\lambda)Q_1(\lambda - \pi) + \Omega F_1(\lambda + 2\pi/3)Q_1(\lambda - \pi/3) + \Omega^2 F_1(\lambda - 2\pi/3)Q_1(\lambda + \pi/3) = 0, \tag{60}$$

where

$$\Omega = -t_0^+ (\Omega^3 = 1) \qquad \text{and} \qquad F_1(\lambda) = \prod_{l\in I} f_l(\lambda). \tag{61}$$

On the other hand the solution (36) for $T^{\pm}(\lambda)$ brings (25) into a similar functional equation:

$$F_2(\lambda)Q_2(\lambda - \pi) + \Omega^2 F_2(\lambda + 2\pi/3)Q_2(\lambda - \pi/3) + \Omega F_2(\lambda - 2\pi/3)Q_2(\lambda + \pi/3) = 0, \tag{62}$$

where

$$F_2(\lambda) = \sin^L\lambda\prod_{m\in\bar{I}} f_m(\lambda). \tag{63}$$

Let us consider the case where $L \neq 3n$. In this case since $P = (2\pi/L)j$ ($j = 0, \ldots, L-1$), equation (47) gives $t_0^- = 1$, and consequently $\Omega = 1$ in (60) and (62).

We intend to argue now that there exist pairs $\{Q_1(\lambda), Q_2(\lambda)\}$ satisfying (60) and (62) with $\Omega = 1$ which lead to 'physical' solutions for the nested BAE (3), i.e. solutions which are inside the usual bounds $n_0 \leqslant n_1 \leqslant n_2$ or equivalently

$$\deg Q_1 \leqslant \deg Q_2 - \deg Q_1 \leqslant L - \deg Q_2. \tag{64}$$

First of all, we have special solutions for (60) and (62) which can be written as

$$Q_1(\lambda) = Q_{1spec}(\lambda) = \prod_{m\in\bar{I}} f_m(\lambda+\pi), \qquad Q_2(\lambda) = Q_{2spec}(\lambda) = \prod_{l\in I} f_l(\lambda+\pi). \tag{65}$$

Let us check these formulae, inserting them into equations (60) and (62). The left-hand side of equation (60) becomes (see (34))

$$\begin{aligned}
\prod_{l\in I} f_l(\lambda) > \prod_{m\in\bar{I}} f_m(\lambda) &+ \prod_{l\in I} f_l(\lambda+2\pi/3) \prod_{m\in\bar{I}} f_m(\lambda+2\pi/3) \\
&+ \prod_{l\in I} f_l(\lambda-2\pi/3) \prod_{m\in\bar{I}} f_m(\lambda-2\pi/3) \\
&= \sin^L(\lambda+\pi/3) - \sin^L(\lambda-\pi/3) + \sin^L(\lambda+\pi) \\
&- \sin^L(\lambda+\pi/3) + \sin^L(\lambda-\pi/3) - \sin^L(\lambda-\pi) = 0.
\end{aligned}$$

Similarly the left-hand side of equation (62) becomes

$$\begin{aligned}
&\sin^L(\lambda)(\sin^L(\lambda+\pi/3) - \sin^L(\lambda-\pi/3)) + \sin^L(\lambda+2\pi/3)(\sin^L(\lambda+\pi) - \sin^L(\lambda+\pi/3)) \\
&\qquad + \sin^L(\lambda-2\pi/3)(\sin^L(\lambda-\pi/3) - \sin^L(\lambda-\pi)) = 0.
\end{aligned}$$

Let $0 \leqslant \mathcal{I} \leqslant L$ be the number of elements of the set I, then the degrees of these special solutions Q_1 and Q_2 are $L - \mathcal{I}$ and $\mathcal{I}$ respectively. Inequalities (64) for these pairs become

$$L - \mathcal{I} \leqslant 2\mathcal{I} - L \leqslant L - \mathcal{I}, \tag{66}$$

which is the same as the equality $2L = 3\mathcal{I}$. This is not enough for our purposes, especially for $L \neq 3n$, so we have to look for additional solutions. These do exist due to the fact that the matrices in equations (28) and (29) for Q_1 and Q_2 have rank 1.

According to the analysis of functional equations of type (60) or (62) made in some previous papers [10] it was noticed that equations of this type have some conjectured properties that we are going to accept. If in (60) or (62) $F_i(\lambda)$ ($i = 1, 2$) have a trigonometric form $F_i(\lambda) = \prod_{j=1}^N \sin(\lambda - a_j)$, of degree N, in general there exists a trigonometric solution of the form $Q_i(\lambda) = \prod_{j=1}^m \sin(\lambda - b_j)$ of degree m. This degree depends on the value of Ω appearing in the equation. In particular if $\Omega = 1$ then $m = N/2 + 1$ for N even and $m = (N-1)/2$ for N odd. Only for special choices of $F_i(\lambda)$ can this degree be decreased. We call these solutions Q_{1gen}, Q_{2gen} general ones.

Due to (61) and (63) we have $\deg F_1(\lambda) = \mathcal{I}$ and $\deg F_2(\lambda) = 2L - \mathcal{I}$. If we choose $\mathcal{I}$ even then $2L - \mathcal{I}$ is also even and the equations (60) and (62) have trigonometric solutions for Q_1 and Q_2, with $\deg Q_1 = \mathcal{I}/2 + 1$ and $\deg Q_2 = (2L - \mathcal{I})/2 + 1$. On the other hand for odd values of $\mathcal{I}$ we have $\deg Q_1 = (\mathcal{I} - 1)/2$ and $\deg Q_2 = (2L - \mathcal{I} - 1)/2$.

Before considering arbitrary values of L let us restrict ourselves initially to the particular case $L = 7$. In table 1 we list the predicted degrees of the sine polynomials Q_1 and Q_2. We underline pairs Q_1, Q_2 which satisfy the inequalities (64) and in the last column of this table we present the eigensectors where we expect to find the predicted simple energy levels.

First of all we see that only odd $\mathcal{I}$ leads to a 'physical' solution. This fact is consistent with the results of our 'experimental' observations formulated in conjecture 1.

Table 1. Degrees of the polynomials Q_1 and Q_2 from the possible solutions for $L+7$. The special solutions Q_{1spec} and Q_{2spec} are given by (65) and the general ones Q_{1gen} and Q_{2gen} are discussed in the text.

$\mathcal{I}$	deg Q_{1gen}	deg Q_{2spec}	deg Q_{1spec}	deg Q_{2gen}	sector
0	1	0	7	8	—
1	$\underline{0}$	$\underline{1}$	6	6	(0, 1, 6)
2	2	2	5	7	—
3	$\underline{1}$	$\underline{3}$	4	5	(1, 2, 4)
4	3	4	3	6	—
5	2	5	$\underline{2}$	$\underline{4}$	(2, 2, 3)
6	4	6	1	5	—
7	3	7	$\underline{0}$	$\underline{3}$	(0, 3, 4)

We see further that for small $\mathcal{I}$ the 'physical' solution is a pair consisting of a general solution Q_{1gen} and a special one Q_{2spec}. For odd $\mathcal{I} = 2k+1$ we have deg $Q_{1gen} = (\mathcal{I}-1)/2 = k$ and deg $Q_{2spec} = \mathcal{I} = 2k+1$. Inserting these formulae into inequalities (64) we obtain $k \leqslant k+1 \leqslant L-2k-1$. For $L = 3n+l$ ($l = 1, 2$) we obtain the upper boundary for k:

$$k \leqslant n + \frac{l-2}{3}. \tag{67}$$

On the other side for $\mathcal{I}$ large enough we combine a special solution Q_{1spec}, which has degree $L-\mathcal{I} = L-2k-1$, and a general one Q_{2gen}, which has degree $(2L-\mathcal{I}-1)/2 = L-k-1$. Inequalities (64) become $L-2k-1 \leqslant k \leqslant k+1$. Taking $L = 3n+l$ ($l = 1, 2$) we obtain now the lower boundary for k:

$$k \geqslant n + \frac{l-1}{3}. \tag{68}$$

There are no holes between (67) and (68), so we have a 'physical' solution for every odd $\mathcal{I}$ and the corresponding energy levels have to be in sector $(k, k+1, L-2k-1)$. This explains conjecture 1!

The case $L = 3n$ is more complicated and we did not derive similar results.

6. Summary and conclusions

Although the exact integrability is a property independent of the lattice size, the exact solutions of the associated BAEs for finite chains were known in very few cases. The XXZ chain at the special value of the anisotropy $\Delta = (q+q^{-1})/2$, $q = \exp(i2\pi/3)$, is one of these examples. Motivated by this result we made extensive numerical calculations for the $SU(3)$ generalization of the XXZ chain, namely the $SU(3)$ Perk–Schultz model, also at the special anisotropy $q = \exp(i2\pi/3)$. Surprisingly, as we stated in section 3, the numerical results reveal that many of the eigenenergies (not all of them) are expressed as combinations of simple cosines and, apart from some selection rules, are quite similar to the energies of a free-fermion chain (or XXZ at $\Delta = 0$).

Our numerical results indicate the five conjectures presented in section 3. The first two conjectures concern the periodic quantum chain and give the exact expression for the energy and momentum of several eigenfunctions. In several sectors the lowest energy value is also predicted. In order to explain these results analytically we present in section 5 a set of BAE solutions that are consistent with the conjectured energies. However the set of solutions we obtained is larger then the conjectured one. This implies that some of our solutions, although having non-coinciding roots, are unphysical, corresponding to a zero vector, since the associated energy is missing from the eigenspectrum. These missing BAE solutions appear in the sectors (n_0, n_1, n_2) not satisfying the bound $n_0 \leqslant n_1 \leqslant n_2 \leqslant 2L/3$. From the functional relations derived in section 4 we were able to explain at least for the cases $L \neq 3n$ the selection rules appearing in conjecture 1. In the case $L = 3n$ the degrees of the trigonometric solutions of the functional equations are more difficult to predict and we could not explain conjecture 1.

The last three conjectures concern the eigenspectra of the Hamiltonian with the quantum symmetry $SU(3)_q$, i.e. the free-boundary case. These conjectures show no selection rules, in contrast with the periodic case. Again in this case we present a set of solutions of the BAEs sharing the same energies as those of conjecture 3. The functional relations in this case are more complicated and we leave this analysis for a future work.

Finally it is interesting to mention that the finite-size corrections obtained from the conjectured eigenenergies of section 3 give us some conformal dimensions of the underlying conformal field theory (CFT) governing the large-distance physics of the model. As a consequence of the conformal invariance of the infinite system these eigenenergies [13] should behave as

$$E = e_\infty L + \frac{\pi}{6L} v_s (12x_o - c) + \mathrm{o}(L^{-1}), \tag{69}$$

in the periodic case, and

$$E = e_\infty L + f_s + \frac{\pi}{24L} v_s (24x_o^s - c) + \mathrm{o}(L^{-1}), \tag{70}$$

in the open-boundary cases. In the above expression e_∞ and f_s are the energy per site and surface energy in the bulk limit, v_s is the sound velocity, c is the central charge and x_o, x_o^s are the conformal dimensions governing the power-law decay of correlations in the periodic and open-chain cases.

In the periodic case, conjecture 2 gives the asymptotic behaviour for the lowest eigenenergies

$$E = e_\infty L - \frac{\pi}{6L} v_s (-2) + \mathrm{O}(L^{-3}) \qquad \text{for } L = 3n, \tag{71}$$

$$E = e_\infty L - \frac{\pi}{6L} v_s \frac{2}{3} + \mathrm{O}(L^{-3}) \qquad \text{for } L \neq 3n, \tag{72}$$

where $e_\infty = -(2/3 + \sqrt{3}/\pi)$ and $v_s = \sqrt{3}$ can be inferred from the lowest eigenenergy with momentum $2\pi/L$ of conjecture 1.

The underlying $U(1) \otimes U(1)$ CFT governing these quantum chains is expected to have a central charge $c = 2$ and when formulated in the torus should have the conformal dimensions [14]

$$x(n_1, n_2; m_1, m_2) = x_p(n_1^2 - n_1 n_2 + n_2^2) + \frac{1}{12x_p}(m_1^2 + m_1 m_2 + m_2^2), \tag{73}$$

where x_p is related to the compactification ratio and n_1, n_2, m_1, m_2 are expected to be integers. Assuming $c = 2$ in (69) and comparing with relations (71) and (72) we obtain for the predicted lowest eigenenergies the associated dimensions $x = 1/3$ for $L = 3n$ and $x = 1/9$ for $L \neq 3n$. From (73) these dimensions can be identified with $x(1, 1; 0, 0) = x_p = 1/3$ and $x(1/3, -1/3; 0, 0) = x_p/3 = 1/9$, by taking $x_p = \eta/2\pi = 1/3$. The fractional values in the last case occur because the ground state for lattices with sizes $L \neq 3n$ does not represent, in the bulk limit, the true vacuum of the CFT, since it contains topological defects.

In the case of open boundaries conjecture 5 gives us for the ground state

$$E = e_\infty L + f_s - \frac{\pi}{24L} v_s (-2) + \mathrm{O}(L^{-3}), \tag{74}$$

where e_∞ and v_s have already been obtained in the periodic case and $f_s = 3/2$. Comparing (74) with (70) we obtain $c = -2$. This can be understood since the quantum chain with open boundaries is $SU(3)_q$ symmetric, with $q = \mathrm{e}^{\mathrm{i}\eta}$, $\eta = 2\pi/3$, and the expected [15] conformal anomaly in this case is $c = 2 - 24/m(m+1)$, where $m = \eta/(\pi - \eta) = 2$. Similar analysis can also be performed for the excited states.

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